SELF-CONSISTENT FIELD METHOD IN THE PROBLEM ABOUT THE EFFECTIVE PROPERTIES OF AN ELASTIC COMPOSITE

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This paper is devoted to the calculation of effective elastic properties of a medium containing a random field of ellipsoidal inhomogeneities. It is assumed that the centers of the inclusions (the inhomogeneities) form a random spatial lattice, i.e., the field of inhomogeneities considered is strongly correlated. The interaction between the inhomogeneities is taken into account within the framework of the self-consistent field approximation. It hence turns out that the symmetry of the tensor of the elastic properties of the medium is determined by the symmetry of the elastic properties of the inclusion matrix, as well as by the symmetry of the spatial lattice formed by the mathematical expectations of the centers of the inclusions.

The construction of a tensor of the effective elastic properties of a medium containing a random field of inhomogeneities is related to the solution of the problem about the interaction of many particles. The self-consistent field method is one of the widespread methods of solving such problems. A whole series of papers can be mentioned in which this method was used to determine the effective properties of different inhomogeneous media: polycrystalline [1, 2], composite [3], and in problems about wave propagation in media with defects [4, 5]. Self-consistent field reasoning was used in [6] to construct a successive approximations procedure when the solution of the problem is refined by interaction at each step. The essential constraint of the proposed scheme for application of the self-consistent field method, which is substantially identical for all the papers listed, is the impossibility of examining strongly correlated fields of inhomogeneities. The assumption of a uniform defect distribution or of a weak correlation between remote points for a random field of elastic properties is always taken as the keystone, here.

The problem under consideration was investigated in [7-9] by random function theory methods. If it is impossible to neglect the correlation scale of the random field of inhomogeneities, the application of these methods also does not yield visible results. A rigorous and relatively simple solution is obtained successfully only under the assumption of "strong isotropy" of the medium, which again eliminates the case of regularly disposed inhomogeneities [7].

The utilization of self-consistent field reasoning in this paper permits being strained by just two-point correlation functions in statistical averaging, which affords the possibility of obtaining a solution (in closed form) even for strongly correlated fields of inhomogeneities. A medium is considered in which the field of ellipsoidal inhomogeneities forms a random space lattice. For a low concentration of inclusions, when their interaction can be neglected, the solution obtained agreees with the Hill solution [3]. When interaction is taken into account in a first approximation, the symmetry of the tensor of the effective elastic properties of the medium turns out to depend not only on the symmetry of the tensors of the elastic properties of the matrix and the inclusions, but also on the symmetry of the lattice formed by the inclusions.

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1. As is known, the self-consistent field method is based on solution of the problem of an isolated particle in an arbitrary external field. Hence, let us turn initially to a consideration of a single ellipsoidal inhomogeneity in an unbounded homogeneous elastic medium.

Let L_0 be a constant tensor of the elastic moduli of the fundamental medium, and let $L_0 + L_1$ be the same for an ellipsoidal inhomogeneity occupying a domain V whose characteristic function is $\Theta(s)$ (s is a point of the medium with the radius vector **r**; $\Theta(s)=1$ for $s \in V$ and $\Theta(s)=0$ for $s \overline{\in} V$). Let $\varepsilon_0(\sigma_0)$ denote the continuous external strain (stress) field which would exist for $\mathbf{L}_1=0$.

Let us represent the field of the displacement vector **u** in a medium with an inclusion as

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_1, \ \mathbf{u}_1(s) = -\int \nabla U(s', s) \cdot \mathbf{L}_0 \cdot \boldsymbol{\mu}(s') \Theta(s')\, dV', \tag{1.1}$$

where $\mathbf{u}_0$ is the displacement corresponding to the external field, $\mathbf{u}_1$ is a perturbation caused by the inhomogeneity, U(s', s) is the Green's tensor of the fundamental medium, μ(s) can be treated as the density of dislocation moments induced by the external field in the domain V by using the results of continual dislocation theory [10], which simulate the inhomogeneity, and the log denotes convolution of the tensors with respect to two subscripts.

The selection of the solution in the form (1.1) automatically satisfies the conditions at infinity. Applying the operator $\nabla\cdot(\mathbf{L}_0\cdot\nabla)$ to the vector field **u** and using the properties of the Green's tensor, we obtain

$$\nabla\cdot(\mathbf{L}_0\cdot\nabla\mathbf{u})=\nabla\cdot(\mathbf{L}_0\cdot\boldsymbol{\mu}\Theta). \tag{1.2}$$

Hence, if $\boldsymbol{\mu}=-\mathbf{G}_0\cdot\mathbf{L}_1\cdot\nabla\mathbf{u};\ (\mathbf{G}_0=\mathbf{L}_0^{-1})$, then the vector **u** (1.1) is a solution of the problem of an ellipsoidal inhomogeneity in an unbounded elastic anisotropic medium.* Substituting (1.2) into (1.1) for $\mathbf{u}_1$ and then taking the operator def (the symmetrized gradient) of both sides, we arrive at an equation for the field tensor $\varepsilon_1=\operatorname{def}\mathbf{u}_1$:

$$\boldsymbol{\varepsilon}_1(\mathbf{r}) + \int \mathbf{K}(\mathbf{R})\cdot\mathbf{L}_1\cdot\boldsymbol{\varepsilon}_1(\mathbf{r}')\Theta(\mathbf{r}')\,dV' = -\int \mathbf{K}(\mathbf{R})\cdot\mathbf{L}_1\cdot\boldsymbol{\varepsilon}_0(\mathbf{r}')\Theta(\mathbf{r}')\,dV'. \tag{1.3}$$

The components of the tensor K (R) are $K_{ijkl}(\mathbf{R}) = -[\nabla_k\nabla_l U_{ij}(\mathbf{R})]_{(ik)(jl)}$, $\mathbf{R}=\mathbf{r}'-\mathbf{r}$ (the parentheses indicate symmetrization with respect to the corresponding subscripts).

The kernel K(R) of the integral operator in (1.3) is expressed in terms of the second derivatives of the Green's tensor **U**. On continuous tensor-functions $\boldsymbol{\Phi}$ such that

$$\int\limits_{|\mathbf{R}|>1} \mathbf{K}(\mathbf{R})\cdot\boldsymbol{\Phi}(\mathbf{r}')\,dV' < \infty,$$

this operator can be defined by the following formula [11, 12]:

$$\int \mathbf{K}(\mathbf{r}'-\mathbf{r})\cdot\boldsymbol{\Phi}(\mathbf{r}')\,dV' = \int \mathbf{K}(\mathbf{C}^{-1}\xi-\mathbf{r})\cdot\boldsymbol{\Phi}(\mathbf{C}^{-1}\xi)\,|\mathbf{C}^{-1}|\,dV_\xi + \mathbf{A}\cdot\boldsymbol{\Phi}(\mathbf{r}), \quad \xi=\mathbf{C}\mathbf{r}', \ |\mathbf{C}^{-1}|=\det\mathbf{C}^{-1}. \tag{1.4}$$

Here **C** is the tensor of an affine transformation which takes the region **V** over into a unit sphere. The constant tensor A equals

$$\mathbf{A} = \frac{1}{4\pi}\int\limits_{(\Gamma_1)} \widetilde{\mathbf{K}}(\mathbf{C}^{-1}\mathbf{k})\,d\Gamma, \tag{1.5}$$

where $\widetilde{\mathbf{K}}(\mathbf{k})$ is a Fourier transformation of the kernel K(R), and Γ_1 is the surface of a unit

*The tensor μ is defined to the accuracy of the component μ_0 for which $\nabla\cdot(\mathbf{L}_0\,\boldsymbol{\mu}_0)=0$. Treating μ as a dislocation moment density together with the conditions $\varepsilon=(\mathbf{L}_0+\mathbf{L}_1)\cdot\sigma$ for $s\in V$ and $\varepsilon=\mathbf{L}_0\cdot\sigma$ for $s \overline{\in} V$ yields $\mu_0=0$.

sphere in k space. The integral in the right side of (1.4) is understood in the Cauchy principal-value sense.

Treating the vector potential $\mathbf{u}_1$ as a displacement from some dislocation moments distribution permits determination of the operator with kernel $\mathbf{K}(\mathbf{R})$ on constant bivalent tensors $\mathbf{\Phi}_0$.

In fact, the integral which diverges formally for $R=0$ and $R \to \infty$ $(R=|\mathbf{R}|)$

$$\int \mathbf{K}(\mathbf{R}) \cdot \mathbf{\Phi}_0 dV' = -\operatorname{def} \int \nabla \mathbf{U}(\mathbf{R}) \cdot \mathbf{\Phi}_0 dV' \tag{1.6}$$

can be given meaning by considering it as the expression for the total strain of the medium when a dislocation moments field of density $\mu = \mathbf{G}_0 \cdot \mathbf{\Phi}_0$ is present therein. Such a dislocation moments distribution causes a strain, whose "plastic" part agrees with the symmetric part of the tensor μ [10, 13]. If the strain is not constrained at infinity (this case will interest us later), then internal stresses in the medium will hence be lacking and the total strain agrees with the "plastic" strain. This affords the possibility of defining regularization of the integral (1.6) by the formula

$$\int \mathbf{K}(\mathbf{R}) \cdot \mathbf{\Phi}_0 dV = \mathbf{G}_0 \cdot \mathbf{\Phi}_0. \tag{1.7}$$

Equation (1.4) has been obtained in [12], where the following theorem about polynomial conservativeness was used for the proof. If the external field ε_0 is a polynomial of degree m in the neighborhood of an ellipsoidal inclusion, then the field ε within the inclusion is also a polynomial of degree m. In particular, if the field ε_0 is homogeneous (constant), then the field ε_1 is also homogeneous and has the form [12]

$$\varepsilon_1 = \mathbf{\Lambda} \cdot \mathbf{L}_1 \cdot \varepsilon_0; \quad \mathbf{\Lambda} = -\mathbf{A} \cdot [\mathbf{A} + \mathbf{A} \cdot \mathbf{L}_1 \mathbf{A}]^{-1} \cdot \mathbf{A}, \tag{1.8}$$

where the tensor $\mathbf{A}$ is defined by (1.5).

2. Now let us consider a random (homogeneous and ergodic)field of ellipsoidal inhomogeneities in an infinite homogeneous elastic medium. Let $\Theta(s)$ be the characteristic function of the domain V occupied by inclusions of some specific realization. Analogously to the above, let us seek the solution in the form (1.1), where $\mathbf{u}_1$ is the perturbation caused by the presence of the inhomogeneities. Then we obtain an equation for the tensor field $\varepsilon_1 = \operatorname{def} \mathbf{u}_1$ which agrees with (1.3) in form, where $\Theta(s)$ is understood to be the characteristic function of the domain V occupied by the inclusions. The tensor $\mathbf{L}_1$ now depends on the point s, since it will be different for different inclusions in the general case. As has been remarked in [12], (1.3) is an equation for ε_1 within the domain **V**.

The solution is continued uniquely in the domain $\bar{\mathbf{V}}$ (the complement of **V** in the whole space) for known ε_1 within **V**. For the field ε_1 within **V** we have

$$\Theta(s)\varepsilon_1(s) + \int \mathbf{K}(s', s) \cdot \mathbf{L}_1(s') \cdot \varepsilon_1(s')\Theta(s')\Theta(s)\,dV' = -\int \mathbf{K}(s', s) \cdot \mathbf{L}_1(s')\varepsilon_0(s')\Theta(s')\Theta(s)\,dV' \tag{2.1}$$

which is the staring point for the construction of the solution of the problem, the mean with respect to the ensemble of realizations of the random field of inclusions. We obtain such a solution within the framework of the self-consistent field approximation. This means that each of the inclusions of any specific realization will be considered isolated in some equivalent external field $\hat{\varepsilon}$, comprised of an external field ε_0, later assumed homogeneous, and the field induced by the surrounding inclusions.

Let us note that the field $\hat{\varepsilon}$ need not be selected as homogeneous. If the concentration of inclusions is small, so that the field from all the surrounding inhomogeneities varies insignificantly within the volume occupied by a typical inclusion, then $\hat{\varepsilon}$ can be considered a constant tensor. Accepting this assumption, we obtain on the basis of (1.8) that within any inclusion the field ε_1 has the form

$$\varepsilon_1 = \mathbf{\Lambda} \cdot \mathbf{L}_1 \cdot \hat{\varepsilon}, \tag{2.2}$$

within the framework of the self-consistent field approximation, where Λ and L_1 are random tensors.

We obtain the equation for the field $\hat{\varepsilon}$ by substituting (2.2) into (2.1) and averaging the result over the ensemble of realizations of the random field of inclusions

$$\left[\langle\Theta(s)\Lambda(s)\cdot L_1(s)\rangle+\int K(s',s)\cdot\langle L_1(s')\cdot\Lambda(s')\cdot L_1(s')\times\right.$$
$$\left.\times\Theta(s')\Theta(s)\rangle\,dV'\right]\cdot\hat{\varepsilon}=-\int K(s',s)\cdot\langle L_1(s')\Theta(s')\Theta(s)\rangle\,dV'\cdot\varepsilon_0. \tag{2.3}$$

The angular brackets denote the mentioned averaging operation. To construct the means in (2.3), a specific model of the random field of inclusions in the medium must be given. Let us examine a possible model. Let the centers of the inclusions occupy the nodes of the random spatial lattice. Let us assign the triple of random vectors a_1, a_2, a_3 to an elementary cell (Bravais cell) of this lattice. Let us connect the set of lattice nodes to the set of triples of integer subscripts (k, l, m) by defining the vector r_{klm} of a node by the relationship

$$r_{klm}=\operatorname{sign}k\sum_{i=1}^{|k|}a_1^{(i)}+\operatorname{sign}l\sum_{i=1}^{|l|}a_2^{(i)}+\operatorname{sign}m\sum_{i=1}^{|m|}a_3^{(i)}$$
$$(k,l,m=-\infty,\ldots,-1,0,1,\ldots,+\infty). \tag{2.4}$$

Here all the vectors $a_j^{(i)}$ are independent random variables with known distribution functions φ_j $(j=1,2,3)$. The lattice nodes are the centers of inclusions of ellipsoidal shape with random values of the semiaxes c_1, c_2, c_3 and random orientation which the orthogonal tensor Q defines for a fixed basis. The elastic properties of the inclusions are given by the random tensor L_1. The combined distribution functions of all the quantities mentioned will be considered known. The means in (2.3) must be constructed for the described field of inclusions:*

$$\Psi^{(1)}(s',s)=\langle L_1(s')\cdot\Lambda(s')\cdot L_1(s')\Theta(s')\Theta(s)\rangle;$$
$$\Psi^{(2)}(s',s)=\langle L_1(s')\Theta(s')\Theta(s)\rangle;\ \Psi^{(3)}=\langle\Lambda(s)\cdot L_1(s)\Theta(s)\rangle. \tag{2.5}$$

Following the method used in [15], let us temporarily consider the point s' a random, uniformly distributed point in the whole space. Let us introduce the notation s'_* for the random point. Because of the ergodicity the means (2.5) with respect to the ensemble of realizations of the random field of inclusions equal the means with respect to all possible positions of the point s'_* if $\Theta(s)$ is a fixed typical realization of this field. Let Θ_{kpm} be the characteristic function of an inclusion whose center is at a point with a radius-vector r_{kpm}.

Then

$$\Theta(s)=\sum_{k,p,m=-\infty}^{\infty}\Theta_{kpm}(s). \tag{2.6}$$

It is admissible to consider that the ellipsoid, within which is the point s'_*, has the subscripts 000. The dimensions and orientation of the ellipsoid Θ_{000} are random variables whose distribution agrees with the ensembles. Let us consider the mean $\Psi^{(1)}(s',s)$ in (2.5). Substituting (2.6) into the expression for $\Psi^{(1)}(s',s)$, we obtain

$$\Psi^{(1)}(s',s)=\sum_{k,p,m=-\infty}^{\infty}\Psi_{kpm}^{(1)}(s',s)=\sum_{k,p,m=-\infty}^{\infty}\langle\Theta_{000}(s'_*)\,\Theta_{kpm}(s)\,L_1(s'_*)\cdot\Lambda(s'_*)\cdot L_1(s'_*)\rangle. \tag{2.7}$$

*The construction of such means is one of the problems considered in geometric probability theory. The fundamental principles are elucidated in [14]. A number of results for the one-dimensional case have been obtained in [15].

(The vector R connecting the points s'_* and s is fixed.)

Let us consider the component corresponding to k = p = m = 0 in (2.7)

$$\Psi^{(1)}_{000} = \langle \Theta_{000}(s'_*) \cdot \Theta_{000}(s)\, \mathbf{L}_1(s'_*) \cdot \Lambda(s'_*) \cdot \mathbf{L}_1(s'_*) \rangle, \tag{2.8}$$

where $\Psi^{(1)}_{000}$ is the mean with respect to all positions of the point s'_* when s'_* and s turn out to be within one inclusion. The mean (2.8) can be represented as

$$\Psi^{(1)}_{000} = \left\langle \mathbf{L}_1 \cdot \Lambda \cdot \mathbf{L}_1 \frac{1}{V_0} J(\mathbf{R}, c_1, c_2, c_3, \mathbf{Q}) \right\rangle, \tag{2.9}$$

where $J(\mathbf{R}, c_1, c_2, c_3, \mathbf{Q})$ is the volume within the ellipsoidal domain occupied by the inclusion (with semiaxes c_1, c_2, c_3 and orientation Q). Incidence of the point s'_* within this volume assures incidence of the point s in the same ellipsoid; V_0 is the mean volume per inclusion. By the affine transformation $\mathbf{C}(\xi = \mathbf{C}\mathbf{R})$, carrying the given ellipsoid over into a unit sphere, the function $J(\mathbf{R}, c_1, c_2, c_3, \mathbf{Q})$ is taken into a spherically symmetric function

$$J(\mathbf{C}^{-1}\xi) = J'(|\xi|). \tag{2.10}$$

It is evident that

$$J(0, c_1, c_2, c_3, \mathbf{Q}) = v_c,$$

where $v_c = (4\pi/3)\, c_1, c_2, c_3$ is the volume of an ellipsoid with the semiaxes c_1, c_2, c_3.

Now let us find the remaining components in the sum (2.7). Let Ω denote the domain representing the ellipsoid Θ_{kpm}, whose center is at the point s. Let $\mathbf{r}'$ be the radius-vector of the point s'_* relative to the center of the ellipsoid Θ_{000}. Then $\Psi^{(1)}_{kpm}(s', s)$ is represented as

$$\Psi^{(1)}_{kpm} = \left\langle \mathbf{L}_1 \cdot \Lambda \cdot \mathbf{L}_1 \frac{1}{V_0} \int \Theta_{000}(\mathbf{r}')\, dV' \right\rangle \left\langle \int\limits_{\Omega(\mathbf{r}'+R)} \gamma_{kpm}(\mathbf{r})\, dV \right\rangle, \tag{2.11}$$

where $\mathbf{r}' + \mathbf{R}$ is the radius-vector of the center of the domain Ω; and γ_{kpm} is the distribution function of the random vector $\mathbf{r}_{kpm}$. Let the variance of this random variable be large compared to the size of the inhomogeneity so that $\gamma_{kpm}(\mathbf{r})$ varies insignificantly within the limits of the domain occupied by a typical inclusion. Applying the theorem of the mean, we obtain

$$\int\limits_{\Omega} \gamma_{kpm}(\mathbf{r})\, dV \simeq \gamma_{kpm}(\mathbf{r}' + \mathbf{R})\, v_c. \tag{2.12}$$

For a small concentration of inclusions $p = \langle v_c \rangle / V_0$ and smallness of the variance as compared with the spacing between inclusions, it can be considered that*

$$\gamma_{kpm}(\mathbf{r}t + \mathbf{R}) \approx \gamma_{kpm}(\mathbf{R}). \tag{2.13}$$

Substituting (2.12). (2.13) in (2.11), we have

$$\Psi^{(1)}_{kpm}(\mathbf{R}) = \gamma_{kpm}(\mathbf{R})\, p \langle \mathbf{L}_1 \cdot \Lambda \cdot \mathbf{L}_1 v_c \rangle. \tag{2.14}$$

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*The assumptions (2.12) and (2.13) are also equivalent to the replacement of the inclusions by some effective dipoles.

Taking account of (2.14), the expression for the mean $\Psi^{(1)}(s', s)$ in (2.7) becomes

$$\Psi^{(1)}(\mathbf{R}) = \Psi^{(1)}_{000}(\mathbf{R}) + p\mathrm{S}(\mathbf{R})\langle \mathbf{L}_1 \cdot \mathbf{\Lambda} \cdot \mathbf{L}_1 v_c \rangle, \tag{2.15}$$

where

$$S(\mathbf{R}) = \sum_{k,p,m=-\infty}^{\infty}{}' \gamma_{kpm}(\mathbf{R}). \tag{2.16}$$

The prime on the summation sign denotes omission of the component k = p = m = 0. Since the random vector $\mathbf{r}_{kpm}$ is the sum of independent random variables, the characteristic function $\widetilde{\gamma}_{kpm}(\mathbf{k})$ of the vector is

$$\widetilde{\gamma}_{kpm}(\mathbf{k}) = \widetilde{\varphi}_1^{|k|}(\operatorname{sign} k\mathbf{k})\,\widetilde{\varphi}_2^{|p|}(\operatorname{sign} p\mathbf{k})\,\widetilde{\varphi}_3^{|m|}(\operatorname{sign} m\mathbf{k}),$$

where $\widetilde{\varphi}_i(\mathbf{k})$ is the characteristic function of the random vector $\mathbf{a}_i$. Passing to the distribution function and substituting the result into (2.16), we obtain after taking the sum

$$S(\mathbf{R}) = \frac{1}{(2\pi)^3}\int \widetilde{S}(\mathbf{k}) e^{(-i\mathbf{k}\mathbf{R})}\, dV_k,$$

$$\widetilde{S}(\mathbf{k}) \prod_{i=1}^{3}\left\{\frac{1-\widetilde{\varphi}_i(\mathbf{k})\widetilde{\varphi}_i(-\mathbf{k})}{[1-\widetilde{\varphi}_i(\mathbf{k})][1-\widetilde{\varphi}_i(-\mathbf{k})]} - 1 + \frac{(2\pi)^3}{V_0}\delta(\mathbf{k}).\right. \tag{2.17}$$

Here $\delta(\mathbf{k})$ is the delta function concentrated at the point $\mathbf{k}=0$.

In particular, if each of the vectors a of the Bravais cell corresponding to the lattice of inhomogeneities is distributed normally with the mathematical expectation $\overline{\mathbf{a}}_i$ and identical variance σ for all the vectors, then $S(\mathbf{k})$ becomes

$$S(\mathbf{k}) = [1-\exp(-\sigma^2 k^2)] \prod_{i=1}^{3}\left[1 - 2\exp\left(-\frac{1}{2}\sigma^2 k^2\right)\cos(a_i k) + \exp(-\sigma^2 k^2)\right]^{-1} - 1 + \frac{(2\pi)^3}{V_0}\delta(\mathbf{k}). \tag{2.18}$$

We obtain analogously for the means $\Psi^{(2)}(s', s)$ and $\Psi^{(3)}$ in (2.5)

$$\Psi^{(2)}(\mathbf{R}) = \Psi^{(2)}_{000}(\mathbf{R}) + pS(\mathbf{R})\langle v_c \mathbf{L}_1 \rangle. \tag{2.19}$$

Here

$$\Psi^{(2)}_{000}(\mathbf{R}) = \langle \mathbf{L}_1 \frac{1}{V_0} J(\mathbf{R}, c_1, c_2, c_3, \mathbf{Q})\rangle; \tag{2.20}$$

$$\Psi^{(3)} = \langle \mathbf{\Lambda} \cdot \mathbf{L}_1 \frac{v_c}{V_0} \rangle. \tag{2.21}$$

Taking account of (2.15) and (2.19)-(2.21), Eq. (2.3) for the equivalent field $\hat{\varepsilon}$ becomes

$$\left[\Psi^{(3)} + \int \mathbf{K}(\mathbf{R}) \cdot \Psi^{(1)}_{000}(\mathbf{R})\, dV + p\int \mathbf{K}(\mathbf{R}) S(\mathbf{R})\, dV \cdot \langle \mathbf{L}_1 \cdot \mathbf{\Lambda} \cdot \mathbf{L}_1 v_c \rangle\right] \cdot$$
$$\cdot \hat{\boldsymbol{\varepsilon}} = -\left[\int \mathbf{K}(\mathbf{R}) \cdot \Psi^{(2)}_{000}(\mathbf{R})\, dV + p\int \mathbf{K}(\mathbf{R}) S(\mathbf{R})\, dV \cdot \langle \mathbf{L}_1 v_c \rangle\right] \cdot \boldsymbol{\varepsilon}_0. \tag{2.22}$$

Let us evaluate the integrals which enter here. From (2.9) and (2.10) we will have

$$\int \mathbf{K}(\mathbf{R}) \cdot \Psi^{(1)}_{000}(\mathbf{R})\, dV = \langle \frac{1}{V_0}\int \mathbf{K}\,(\mathbf{C}^{-1}\boldsymbol{\xi})\, J'(|\boldsymbol{\xi}|)\, |\mathbf{C}^{-1}|\, dV_\xi \cdot \mathbf{L}_1 \cdot \mathbf{\Lambda} \cdot \mathbf{L}_1 \rangle.$$

Using the regularization (1.4), we arrive at the relationship

$$\int \mathbf{K}(\mathbf{R})\,\Psi^{(1)}_{000}(\mathbf{R})\,dV = \langle \frac{v_c}{V_0}\mathbf{A}\cdot\mathbf{L}_1\cdot\mathbf{A}\cdot\mathbf{L}_1\rangle; \tag{2.23}$$

analogously,

$$\int \mathbf{K}(\mathbf{R})\,\Psi^{(2)}_{000}(\mathbf{R})\,dV = \langle \frac{v_c}{V_0}\mathbf{A}\cdot\mathbf{L}_1\rangle. \tag{2.24}$$

In evaluating the integral $\int \mathbf{K}(\mathbf{R})\,S(\mathbf{R})\,dV$ let us note that the inverse transform of the Fourier member with the δ-function in the expression for $S(\mathbf{k})$ yields a constant component. Using the regularization (1.6) and the Parseval equality, we obtain

$$\int \mathbf{K}(\mathbf{R})\,S(\mathbf{R})\,dV = \frac{1}{V_0}(\mathbf{H}+\mathbf{G}_0), \tag{2.25}$$

$$\mathbf{H} = \frac{V_0}{(2\pi)^3}\int \widetilde{\mathbf{K}}(\mathbf{k})\,\widetilde{S}(\mathbf{k})\,dV_k, \tag{2.26}$$

and the integral in (2.26) is understood in the sense of the Cauchy principal value. To the accuracy of the symmetrization and commutation operations, the transformation of the Fourier-kernel $\mathbf{K}(\mathbf{R})$ equals $[\mathbf{k}\mathbf{L}_0\mathbf{k}]^{-1}\otimes\mathbf{k}\otimes\mathbf{k}$ ($\otimes$ denotes the tensor product). Hence, the symmetry group of the tensor H is the intersection of the symmetry group of the elastic modulus tensor of the fundamental medium $\mathbf{L}_0$ and the symmetry group of the function $\widetilde{S}(\mathbf{k})$.

As follows from (2.17), this latter agrees with the symmetry group of the spatial lattice which is the mathematical expectation of the random lattice of inclusions being considered. Let the fundamental medium (matrix) be isotropic, and let the mathematical expectation of the lattice of inclusions be a simple cubic lattice. Then the components of the tensor H (2.26) become

$$H_{ijkl} = \frac{1}{2\mu}\left[\left(\beta_0 - 2\frac{\lambda+\mu}{\lambda+2\mu}\beta_1\right)I_{ijkl} - 2\frac{\lambda+\mu}{\lambda+2\mu}(\beta_1\delta_{ij}\delta_{kl}+\beta_2 B_{ijkl})\right], \tag{2.27}$$

where λ and μ are the Lamé parameters of the fundamental medium, $I_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl}+\delta_{ik}\delta_{lj})$ is a unit quadrivalent tensor, B is a quadrivalent tensor possessing the symmetries of the cubic lattice,

$$\mathbf{B} = \frac{1}{a^4}\sum_{i=1}^{3}\left(\bigotimes_1^4\right)\bar{\mathbf{a}}_i,$$

and $\left(\bigotimes_1^4\right)$ denotes quadruple tensor multiplication of a vector by itself. The coefficients $\beta_0, \beta_1, \beta_2$ in (2.27) are defined by the equalities

$$\beta_0 = \frac{V_0}{3}\int S(\mathbf{k})\,dV;\ \beta_1 = \frac{1}{4}\left[\frac{V_0}{3}\int\frac{k_1^4}{k^4}S(\mathbf{k})\,dV+\beta_0\right];\ \beta_2 = \beta_0 - 5\beta_1.$$

The integrals are here understood in the principal-value sense. For $s(\mathbf{k})$ in (2.18), the β_0, and β_1 are written as absolutely convergent integrals

$$\beta_0 = \frac{1}{3}\iiint_{-\infty}^{\infty}\left[\left(1-\exp\left(-\frac{\sigma^2}{a^2}z^2\right)\right)\prod_{i=1}^{3}\left(1-2\exp\left(-\frac{1}{2}\frac{\sigma^2}{a^2}z^2\right)\times\right.\right.$$

$$\times \cos z_i + \exp\left(-\frac{\sigma^2}{a^2} z^2\right)\Bigg)^{-1} - 1\Bigg] dz_1 dz_2 dz_3;$$

$$\beta_1 = \frac{1}{12}\iiint\limits_{-\infty}^{\infty} \frac{z_1^4}{z^4}\Bigg[\left(1 - \exp\left(-\frac{\sigma^2}{a^2} z^2\right)\right)\prod_{i=1}^{3}\Bigg(1 - 2\exp\left(-\frac{1}{2}\frac{\sigma^2}{a^2} z^2\right)\times$$

$$\times \cos z_i + \exp\left(-\frac{\sigma^2}{a^2} z^2\right)\Bigg)^{-1} - 1\Bigg] dz_1 dz_2 dz_3 + \frac{1}{4}\beta_0;$$

$$\left(z = \sqrt{z_1^2 + z_2^2 + z_3^2}\right).$$

Now substituting (2.21), (2.23), (2.24), (2.25) into (2.22) and solving the equation obtained for the tensor $\hat{\varepsilon}$, we will have

$$\hat{\varepsilon} = \mathbf{D}\cdot\varepsilon_0, \tag{2.28}$$

where

$$\mathbf{D} = \left[\mathbf{I} - p\left\langle\frac{v_c}{V_0}\mathbf{A}\cdot\mathbf{L}_1\right\rangle^{-1}\cdot\left\langle\frac{v_c}{V_0}(\mathbf{H} + \mathbf{G}_0)\cdot\mathbf{L}_1\cdot\Lambda\cdot\mathbf{L}_1\right\rangle\right]^{-1}\times$$
$$\times\left[\mathbf{I} - p\left\langle\frac{v_c}{V_0}\mathbf{A}\cdot\mathbf{L}_1\right\rangle^{-1}\cdot\left\langle\frac{v_c}{V_0}(\mathbf{H} + \mathbf{G}_0)\cdot\mathbf{L}_1\right\rangle\right]. \tag{2.29}$$

Here we used the equality $\mathbf{A}\cdot\mathbf{L}_1 + \mathbf{A}\cdot\mathbf{L}_1\cdot\Lambda\cdot\mathbf{L}_1 = -\mathbf{A}\cdot\mathbf{L}_1$, which is verified by using (1.7).

In combination with (2.2), the relationship (2.28) permits determination of the strain within an arbitrary inclusion within the framework of the self-consistent field approximation, upon application of a homogeneous external field ε_0 to a composite material.

3. Let $\langle\varepsilon\rangle$ be the mean strain of a medium with respect to the ensemble of realization of a random field of inclusions, upon application of a random stress field σ_0. The tensor of the effective elastic pliability is determined by the relationship

$$\langle\varepsilon\rangle = \mathbf{G}_e\cdot\sigma_0. \tag{3.1}$$

Within the framework of the self-consistent field approximation, we find the expression for the mean strain of the medium from (1.1), (1.3), and (2.2) by using the averaging and regularization procedures (1.7) described above

$$\langle\varepsilon\rangle = \varepsilon_0 + \langle\varepsilon_1\rangle = \varepsilon_0 - \int \mathbf{K}(\mathbf{R})\cdot\langle(\mathbf{L}_1\cdot\Lambda\cdot\mathbf{L}_1\cdot\hat{\varepsilon} + \mathbf{L}_1\cdot\varepsilon_0)\times$$
$$\times\Theta(\mathbf{r}')\rangle\, dV' = \varepsilon_0 - \mathbf{G}_0\cdot\left\langle\frac{v_c}{V_0}(\mathbf{L}_1\cdot\Lambda\cdot\mathbf{L}_1\cdot\hat{\varepsilon} + \mathbf{L}_1\cdot\varepsilon_0)\right\rangle.$$

Substituting (2.28) here with the equality $\varepsilon_0 = \mathbf{G}_0\sigma_0$ taken into account and comparing the result (3.1), we arrive at the following expression for the tensor of the effective elastic pliability $\mathbf{G}_e$:

$$\mathbf{G}_e = \mathbf{G}_0 - \mathbf{G}_0\cdot\langle\mathbf{L}_1\cdot\Lambda\cdot\mathbf{L}_1\cdot\mathbf{D} + \mathbf{L}_1\rangle\cdot\mathbf{G}_0,$$

where the tensor D has the form (2.29).

Let all the inclusions have the same size, orientation, and elastic properties. Limiting ourselves to the first three members of the expansion of G in a series in the concentration of inclusions p, we obtain

$$G_e = G_0 - pG_0 \cdot (L_1 \cdot A \cdot L_1 + L_1) \cdot G_0 - p^2 G_0 \cdot L_1 \cdot A \cdot A^{-1}(H + G_0)(L_1 \cdot A \cdot L_1 - L_1) \cdot G_0, \quad (3.2)$$

where the tensor H is defined by the relationship (2.26).

The third member in (3.2) can be neglected for a small concentration of inhomogeneities, which corresponds to no interaction between the inclusions ($\hat{\varepsilon} = \varepsilon_0$). Hence, to the accuracy of the notation, the expression for G_e agrees with the Hill result obtained in [3]. Interaction between the defects is taken into account in a first approximation by a member of order p^2 in the expression (3.2) for G_e. The symmetry of this member depends on the symmetry of the space lattice formed by the centers of the inhomogeneities.

In conclusion, let us note that the application of the scheme developed here is apparently valid for not too high values of the concentration of inhomogeneities. The result can be refined by approximation of the equivalent field by a polynomial $\hat{\varepsilon}$ whose coefficients are found from the self-consistency condition and the minimum potential energy condition of the system.

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